

Scaling transition for long-range dependent Gaussian random fields

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Abstract

In [27] we introduced the notion of scaling transition for stationary random fields X on \mathbb{Z}^2 in terms of partial sums limits, or scaling limits, of X over rectangles whose sides grow at possibly different rate. The present paper establishes the existence of scaling transition for a natural class of stationary Gaussian random fields on \mathbb{Z}^2 with long-range dependence. The scaling limits of such random fields are identified and characterized by dependence properties of rectangular increments.

Keywords: scaling transition; long-range dependence; Gaussian random field; operator scaling random field

1 Introduction

Let $X = \{X(t, s); (t, s) \in \mathbb{Z}^2\}$ be a stationary random field (RF) on the lattice \mathbb{Z}^2 , $\gamma > 0$ a given number and $K_{[nx, n^\gamma y]} := \{(t, s) \in \mathbb{Z}^2 : 1 \leq t \leq nx, 1 \leq s \leq n^\gamma y\}$ be a sequence of rectangles whose sides grow at possibly different rate $O(n)$ and $O(n^\gamma)$. Assume that for any $\gamma > 0$ there exist a nontrivial RF $V_\gamma = \{V_\gamma(x, y); (x, y) \in \mathbb{R}_+^2\}$ and a normalization $A_n(\gamma) \rightarrow \infty$ such that

$$A_n^{-1}(\gamma) \sum_{(t,s) \in K_{[nx, n^\gamma y]}} X(t, s) \xrightarrow{\text{fdd}} V_\gamma(x, y), \quad (x, y) \in \mathbb{R}_+^2, \quad n \rightarrow \infty. \quad (1.1)$$

We say that RF X *exhibits scaling transition* if there exists $\gamma_0 > 0$ such that

$$V_\gamma \stackrel{\text{fdd}}{=} V_+, \quad \gamma > \gamma_0, \quad V_\gamma \stackrel{\text{fdd}}{=} V_-, \quad \gamma < \gamma_0 \quad \text{and} \quad V_+ \not\stackrel{\text{fdd}}{=} aV_- \quad (\forall a > 0). \quad (1.2)$$

See the end of this sec. for all unexplained notation. In other words, (1.2) say that the scaling limits V_γ in (1.1) do not depend on γ for $\gamma > \gamma_0$ and $\gamma < \gamma_0$ and are different up to a multiplicative constant (the last condition is needed to exclude a trivial change of the scaling limit by a linear change of normalization). In the sequel, V_{γ_0} will be called the *well-balanced* scaling limit of X , and V_+, V_- the *unbalanced* scaling limits of

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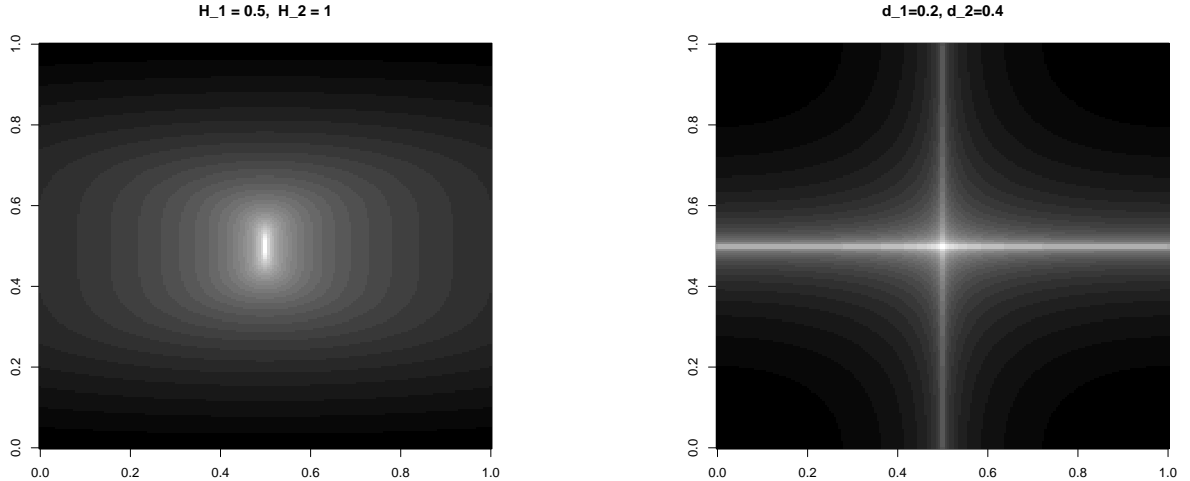
X . Obviously, if the limits $V_\gamma \stackrel{\text{fdd}}{=} V$ in (1.1) are the same for any $\gamma > 0$, the RF X does not exhibit scaling transition.

The notion of scaling transition was introduced in our paper [27], which also established the existence of such transition for a class of aggregated α -stable ($1 < \alpha \leq 2$) RFs on \mathbb{Z}^2 . The last paper identified the scaling limits V_+, V_-, V_{γ_0} and characterized these RFs by certain dependence properties of increments on rectangles $K \subset \mathbb{R}_+^2$.

The present paper extends the results of [27] by proving the existence of scaling transition for a natural class of stationary long-range dependent (LRD) Gaussian RFs with Type I spectral density f_I and its absence for Gaussian RFs with Type II spectral density f_{II} in (1.3):

$$\text{Type I density: } f_I(x, y) = \frac{g(x, y)}{(|x|^2 + c|y|^{2H_2/H_1})^{H_1/2}}, \quad \text{Type II density: } f_{II}(x, y) = \frac{g(x, y)}{|x|^{2d_1}|y|^{2d_2}}, \quad (1.3)$$

where $H_1, H_2 > 0, H_1 H_2 < H_1 + H_2, c > 0, 0 < d_1, d_2 < 1/2$ are parameters and g is a bounded positive function having a positive limit $g(0, 0) > 0$ at the origin (w.l.g., we assume $g(0, 0) = 1$). Type II spectral densities f_{II} in (1.3) include fractionally integrated class $|1 - e^{-ix}|^{-2d_1}|1 - e^{-iy}|^{-2d_2}$ discussed in [5], [17], [16]. Notice that f_I has a unique singularity at $(0, 0)$ while f_{II} is singular on both coordinate axes and factorizes at low frequencies into a product of two functions depending on x and y alone. See Fig.1 below.



Type I spectral density f_I of (1.3), $H_1 = 0.5, H_2 = c = 1$ Type II spectral density f_{II} of (1.3), $d_1 = 0.2, d_2 = 0.4$

Figure 1

The main result of the present paper is Theorem 3.1 which says that for Gaussian RFs with spectral density f_I , scaling transition occurs at $\gamma_0 = H_1/H_2$. It turns out that for such RFs the unbalanced scaling limits V_+ and V_- agree, up to a multiplicative constant, with a fractional Brownian sheet $B_{\mathcal{H}_1, \mathcal{H}_2}$ where at least one of the two parameters $\mathcal{H}_1, \mathcal{H}_2$ equals $1/2$ or 1 . Recall that a fractional Brownian sheet $B_{\mathcal{H}_1, \mathcal{H}_2}$ with parameters

$0 < \mathcal{H}_1, \mathcal{H}_2 \leq 1$ is a Gaussian process on $\bar{\mathbb{R}}_+^2$ with zero mean and covariance function

$$EB_{\mathcal{H}_1, \mathcal{H}_2}(x, y)B_{\mathcal{H}_1, \mathcal{H}_2}(x', y') = (1/4)(x^{2\mathcal{H}_1} + x'^{2\mathcal{H}_1} - |x - x'|^{2\mathcal{H}_1})(y^{2\mathcal{H}_2} + y'^{2\mathcal{H}_2} - |y - y'|^{2\mathcal{H}_2}), \quad (1.4)$$

$(x, y), (x', y') \in \bar{\mathbb{R}}_+^2$. Particularly, for $\mathcal{H}_1 = 1/2$, $B_{\mathcal{H}_1, \mathcal{H}_2}(x, y)$ is a usual Brownian motion in x having independent increments in the horizontal direction and, for $\mathcal{H}_1 = 1$, $B_{\mathcal{H}_1, \mathcal{H}_2}(x, y) = xB_{\mathcal{H}_2}(y)$ is random line in x having shift-invariant (completely dependent) increments in the horizontal direction (see sec.2 for the definitions). The case when \mathcal{H}_2 equals 1/2 or 1 is analogous. One may conclude that the unbalanced limits of the above Gaussian RFs have a very special dependence structure (either independence or extreme ('deterministic') dependence along one of the coordinate axes). By contrast, the well-balanced scaling limit V_{γ_0} is not a fractional Brownian sheet and has dependent but not shift-invariant increments in arbitrary direction on the plane. The dependence properties of rectangular increments are made formal in sec.2 leading to the notion of Type I distributional LRD and isotropic/anisotropic LRD properties for RFs on \mathbb{Z}^2 . As shown in Proposition 3.2, stationary Gaussian RFs with spectral density f_{II} in (1.3) do not exhibit scaling transition since in this case, all scaling limits $V_\gamma, \gamma > 0$ are equal to $B_{\mathcal{H}_1, \mathcal{H}_2}$ with $\mathcal{H}_i = d_i + (1/2), i = 1, 2$ up to a multiplicative constant.

The above mentioned differences in the scaling behavior of Gaussian RFs with spectral densities f_{I} and f_{II} (1.3) are reflected in the scaling behavior of these spectral densities. Indeed, the point $\gamma_0 = H_1/H_2$ at which scaling transition occurs in the case of f_{I} can be characterized as a unique point $\gamma = \gamma_0 > 0$ for which a 'non-degenerated' limit

$$\lim_{\lambda \rightarrow 0} \lambda^{H_1} f_{\text{I}}(\lambda x, \lambda^\gamma y) = (|x|^2 + c|y|^{2H_2/H_1})^{-H_1/2} \quad (1.5)$$

exists, since for $\gamma \neq \gamma_0$ the limit in (1.5) is either zero or 'degenerated', in the sense that it does not depend on y . On the other hand, in the case of f_{II} , a 'non-degenerated' scaling limit $\lim_{\lambda \rightarrow 0} \lambda^{2d_1+2d_2\gamma} f_{\text{II}}(\lambda x, \lambda^\gamma y) = |x|^{-2d_1} |y|^{-2d_2}$ exists for any $\gamma > 0$ and does not depend on γ .

It is of interest to extend the results of this paper in several directions. Paper [18] obtains scaling limits of LRD Gaussian RFs with singular spectral density $f(x, y) = g(x, y)(|x - \mu y|^2 + c|y|^{2H_2/H_1})^{-H_1/2}$ having a general anisotropy axis $x - \mu y = 0, \mu \in \mathbb{R}$ instead of $x = 0$ in f_{I} . Further possibilities include investigation of scaling limits in (1.1) for *nonlinear* instantaneous functions $X(t, s) = G(Y(t, s))$ of stationary Gaussian RFs $Y = \{Y(t, s); (t, s) \in \mathbb{Z}^2\}$ with spectral density f_{I} in (1.3) and *non-Gaussian* moving-average RFs

$$X(t, s) = \sum_{(u, v) \in \mathbb{Z}^2} a(t - u, s - v) \varepsilon(u, v), \quad (t, s) \in \mathbb{Z}^2, \quad (1.6)$$

where $\{\varepsilon(u, v); (u, v) \in \mathbb{Z}^2\}$ is an i.i.d. sequence with zero mean and finite variance, and $a(t, s)$ are deterministic coefficients having the form

$$a(t, s) = \frac{g(t, s)}{(|t|^2 + |s|^{2q_2/q_1})^{q_1/2}}, \quad (t, s) \in \mathbb{Z}^2, \quad (1.7)$$

where $g(t, s), (t, s) \in \mathbb{Z}^2$ are bounded with $\lim_{|t|+|s| \rightarrow \infty} g(t, s) = 1$ and $q_1, q_2 > 0$ satisfy $(q_1 + q_2)/2 < q_1 q_2 < q_1 + q_2$. These conditions guarantee that $\sum_{(t, s) \in \mathbb{Z}^2} |a(t, s)|^2 < \infty, \sum_{(t, s) \in \mathbb{Z}^2} |a(t, s)| = \infty$, hence (1.6) is a well-defined LRD RF. We conjecture that RF X in (1.6) exhibits scaling transition at $\gamma_0 = q_1/q_2$ with V_+, V_-, V_{γ_0}

similar as in Theorem 3.1 and H_1, H_2 related to q_1, q_2 by $H_1 = (2/q_2)(q_1 + q_2 - q_1 q_2), H_2 = (2/q_1)(q_1 + q_2 - q_1 q_2)$. See [23], [14] and Remark 3.3 on a different type of scaling transition in telecommunication models. A challenging task is generalization of our limit results to sums $S_n \gamma(\mathbf{x}) = \sum_{\mathbf{t} \in \prod_{i=1}^\nu [1, n^{\gamma_i x_i}]} X(\mathbf{t})$, $\mathbf{x} = (x_1, \dots, x_\nu) \in \mathbb{R}_+^\nu$, $\gamma = (\gamma_1, \dots, \gamma_\nu) \in \mathbb{R}_+^\nu$ of stationary Gaussian or linear RFs on \mathbb{Z}^ν , $\nu > 2$ with spectral density $f(\mathbf{x})$, $\mathbf{x} \in [-\pi, \pi]^\nu$ similar to f_1 in (1.3) and having ν parameters $H_1 > 0, \dots, H_\nu > 0$. We note that, instead of the single ‘balance condition’ $\gamma_2/\gamma_1 = \gamma_0 = H_1/H_2$ when $\nu = 2$, in higher dimensions $\nu > 2$ there are $\nu(\nu - 1)/2 > 1$ ‘balance conditions’ $\gamma_i/\gamma_j = H_j/H_i$, $i \neq j, 1 \leq i, j \leq \nu$. Depending on which of these ‘balance conditions’ are fulfilled or violated, we may expect different scaling limits of $S_n \gamma(\mathbf{x})$. We plan to explore the case $\nu = 3$ in a forthcoming paper.

The notion of scaling transition for RFs is intrinsically related to the LRD property, which is often identified with unboundedness of spectral density. It is clear that an i.i.d. RF with zero mean and finite variance does not exhibit scaling transition since its all scaling limits $V_\gamma, \gamma > 0$ agree with Brownian sheet $B_{1/2, 1/2}$. A similar fact remains true for weakly dependent stationary RFs satisfying some mixing or other weak dependence conditions. On the other hand, scaling limits of LRD RFs form a very rich class and are extensively studied. See, e.g., [1], [2], [9], [11], [12], [17], [20], [21], [22], [27], [29] and the references therein. Stationary Gaussian RFs form probably the most simple class of LRD RFs, for which the asymptotic scaling theory is well-developed [8]. Nevertheless, we think that our results shed a new angle on Gaussian RFs and spatial LRD.

Notation. In what follows, C denotes a generic constant which may be different at different locations. We write $\xrightarrow{\text{fdd}}$, $\stackrel{\text{fdd}}{=}$, and $\not\stackrel{\text{fdd}}{=}$ for the weak convergence, equality and inequality of finite-dimensional distributions, respectively. $\mathbb{R}_+^\nu := \{(x_1, \dots, x_\nu) \in \mathbb{R}^\nu : x_i > 0, i = 1, \dots, \nu\}$, $\bar{\mathbb{R}}_+^\nu := \{(x_1, \dots, x_\nu) \in \mathbb{R}^\nu : x_i \geq 0, i = 1, \dots, \nu\}$, $\mathbb{R}_+ := \mathbb{R}_+^1$, $\bar{\mathbb{R}}_+ := \bar{\mathbb{R}}_+^1$, $\mathbb{R}_0^2 := \mathbb{R}^2 \setminus \{(0, 0)\}$. $\mathbf{1}(A)$ stands for the indicator function of a set A . All equalities and inequalities between random variables are assumed to hold almost surely.

2 Distributional LRD properties of RFs on \mathbb{Z}^2

This sec. presents the definition of Type I distributional LDR property for RFs on \mathbb{Z}^2 and some related definitions introduced in our paper [27]. It is well-known that partial sum limits, or scaling limits, characterize dependence properties of random processes indexed by \mathbb{Z} . Particularly, let $X = \{X(t); t \in \mathbb{Z}\}$ be a stationary process such that its partial sums tend to a process $V = \{V(x); x \geq 0\}$ in the sense that $n^{-H} \sum_{t=1}^{\lfloor nx \rfloor} X(t) \xrightarrow{\text{fdd}} V(x)$ for some $H > 0$. If the limit process V has dependent increments, the process X is said to have *distributional long memory*, or *distributional LRD* property. The last property (originating to Cox [6]) was introduced in Dehling and Phillips [7] and later used and verified for several classes of LRD processes [19], [15], [24], [26].

Let $\ell = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$ be a line in \mathbb{R}^2 . A line $\ell' = \{(x, y) \in \mathbb{R}^2 : a'x + b'y = c'\}$ is said *perpendicular to* ℓ (denoted $\ell' \perp \ell$) if $aa' + bb' = 0$. Write $(u, v) \prec (x, y)$ (respectively, $(u, v) \preceq (x, y)$), $(u, v), (x, y) \in \mathbb{R}^2$ if $u < x$ and $v < y$ (respectively, $u \leq x$ and $v \leq y$) hold. A *rectangle* is a set $K_{(u,v);(x,y)} := \{(s, t) \in \mathbb{R}^2 : (u, v) \prec (s, t) \preceq (x, y)\}$; $K_{x,y} := K_{(0,0);(x,y)}$. Denote $K_{(u,v);(x,y)} + (z, w) := K_{(u+z,v+w);(x+z,y+w)}$

the rectangle $K_{(u,v);(x,y)}$ shifted by $(z,w) \in \mathbb{R}^2$. We say that two rectangles $K = K_{(u,v);(x,y)}$ and $K' = K_{(u',v');(x',y')}$ are *separated by line ℓ'* if they lie on different sides of ℓ' , in which case K and K' are necessarily disjoint: $K \cap K' = \emptyset$. See Fig. 2.

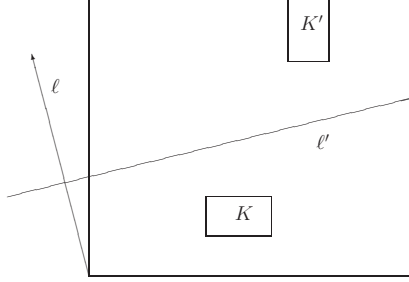


Figure 2

Let $V = \{V(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ be a RF and $K = K_{(u,v);(x,y)} \subset \mathbb{R}_+^2$ be a rectangle. By *increment of V on rectangle K* we mean the difference

$$V(K) := V(x, y) - V(u, y) - V(x, v) + V(u, v).$$

We say that V has *stationary rectangular increments* if

$$\{V(K_{(u,v);(x,y)}); (u, v) \preceq (x, y)\} \stackrel{\text{fdd}}{=} \{V(K_{(0,0);(x-u,y-v)}); (u, v) \preceq (x, y)\}, \quad \text{for any } (u, v) \in \mathbb{R}_+^2. \quad (2.1)$$

Definition 2.1 Let $V = \{V(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ be a RF with stationary rectangular increments, $V(x, 0) = V(0, y) \equiv 0$, $x, y \geq 0$, and $\ell \subset \mathbb{R}^2$ be a given line, $(0, 0) \in \ell$. We say that V has

- (i) independent rectangular increments in direction ℓ if for any orthogonal line $\ell' \perp \ell$ and any two rectangles $K, K' \subset \mathbb{R}_+^2$ separated by ℓ' , increments $V(K)$ and $V(K')$ are independent;
- (ii) invariant rectangular increments in direction ℓ if $V(K) = V(K')$ for any two rectangles $K, K' \subset \mathbb{R}_+^2$ such that $K' = (x, y) + K$ for some $(x, y) \in \ell$;
- (iii) dependent rectangular increments in direction ℓ if neither (i) nor (ii) holds;
- (iv) dependent rectangular increments if V has dependent rectangular increments in arbitrary direction;
- (v) independent rectangular increments if V has independent rectangular increments in arbitrary direction.

Example 2.2 Fractional Brownian sheet $B_{\mathcal{H}_1, \mathcal{H}_2}$ with parameters $0 < \mathcal{H}_1, \mathcal{H}_2 \leq 1$ is a Gaussian process on $\bar{\mathbb{R}}_+^2$ with zero mean and covariance in (1.4). It follows (see [3], Cor.3) that for any rectangles $K = K_{(u,v);(x,y)}$, $K' = K_{(u',v');(x',y')}$

$$\begin{aligned} & \text{EB}_{\mathcal{H}_1, \mathcal{H}_2}(K)B_{\mathcal{H}_1, \mathcal{H}_2}(K') \\ &= \text{E}(B_{\mathcal{H}_1}(x) - B_{\mathcal{H}_1}(u))(B_{\mathcal{H}_1}(x') - B_{\mathcal{H}_1}(u'))\text{E}(B_{\mathcal{H}_2}(y) - B_{\mathcal{H}_2}(v))(B_{\mathcal{H}_2}(y') - B_{\mathcal{H}_2}(v')), \end{aligned} \quad (2.2)$$

where $\{B_{\mathcal{H}_i}(x); x \in \bar{\mathbb{R}}_+\}$ is a fractional Brownian motion on $\bar{\mathbb{R}}_+ = [0, \infty)$ with $\mathbb{E}B_{\mathcal{H}_i}(x)B_{\mathcal{H}_i}(x') = (1/2)(x^{2\mathcal{H}_i} + x'^{2\mathcal{H}_i} - |x - x'|^{2\mathcal{H}_i})$, $i = 1, 2$. (For $\mathcal{H} = 1$, the process $\{B_{\mathcal{H}}(x); x \in \bar{\mathbb{R}}_+\}$ is a random line.) In particular, $B_{\mathcal{H}_1, \mathcal{H}_2}$ has stationary rectangular increments, see ([3], Prop.2). It follows from (2.2) (see [27] for details) that fractional Brownian sheet $B_{\mathcal{H}_1, \mathcal{H}_2}$ has:

- dependent rectangular increments if $\mathcal{H}_i \notin \{1/2, 1\}, i = 1, 2$;
- independent rectangular increments in the horizontal (vertical) direction if $\mathcal{H}_1 = 1/2$ ($\mathcal{H}_2 = 1/2$);
- invariant rectangular increments in the horizontal (vertical) direction if $\mathcal{H}_1 = 1$ ($\mathcal{H}_2 = 1$);
- independent rectangular increments if $\mathcal{H}_1 = \mathcal{H}_2 = 1/2$.

Definition 2.3 Let $X = \{X(t, s); (t, s) \in \mathbb{Z}^2\}$ be a stationary RF. Assume that for any $\gamma > 0$ there exist normalization $A_n(\gamma) \rightarrow \infty$ and a RF $V_\gamma = \{V_\gamma(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ such that (1.1) holds. We say that X has Type I distributional LRD (or X is a Type I RF) if there exists a unique $\gamma_0 \in (0, \infty)$ such that:

- RF V_{γ_0} has dependent rectangular increments; and
- RFs $V_\gamma, \gamma \neq \gamma_0$ do not have dependent rectangular increments; in other words, for each $\gamma \neq \gamma_0, \gamma > 0$ there exists a line $\ell(\gamma)$ such that RF V_γ has either independent, or invariant rectangular increments in direction $\ell(\gamma)$.

Moreover, a Type I RF X is said to have isotropic distributional LRD if $\gamma_0 = 1$ and anisotropic distributional LRD if $\gamma_0 \neq 1$.

Remark 2.1 Let $X = \{X(t, s); (t, s) \in \mathbb{Z}^2\}$ be a stationary RF satisfying (1.1) for some $\gamma > 0$. Then the limit RF V_γ has stationary rectangular increments in the sense of (2.1). Moreover, if $A_n(\gamma) = n^H$ for some $H > 0$, then V_γ satisfies the following self-similarity property (see [27]):

$$\{\lambda V_\gamma(x, y); (x, y) \in \mathbb{R}^2\} \stackrel{\text{fdd}}{=} \{V_\gamma(\lambda^{1/H}x, \lambda^{\gamma/H}y); (x, y) \in \mathbb{R}^2\}, \quad \forall \lambda > 0. \quad (2.3)$$

We note that (2.3) is a particular case of *operator scaling* property for RFs introduced in [4].

3 Main results

This section obtains the presence/absence of scaling transition for stationary Gaussian RFs with spectral densities in (1.3), and a characterization of the dependence properties of their scaling limits. For a given a stationary RF $X = \{X(t, s); (t, s) \in \mathbb{Z}^2\}$ denote

$$S_{n\gamma}(x, y) := \sum_{(t, s) \in K_{[nx, n\gamma y]}} X(t, s), \quad (x, y) \in \mathbb{R}_+^2. \quad (3.1)$$

First, consider a Type I spectral density $f = f_I$ of (1.3):

$$f(x, y) = \frac{g(x, y)}{(|x|^2 + c|y|^{2H_2/H_1})^{H_1/2}}, \quad (x, y) \in \Pi^2 = [-\pi, \pi]^2, \quad (3.2)$$

where $0 < H_1 \leq H_2 < \infty, H_1 H_2 < H_1 + H_2, c > 0$ and g is bounded and continuous at the origin with $g(0, 0) = 1$. We have

$$h(x, y) := \lim_{\lambda \rightarrow 0} \lambda f(\lambda^{1/H_1} x, \lambda^{1/H_2} y) = \frac{1}{(|x|^2 + c|y|^{2H_2/H_1})^{H_1/2}}, \quad (x, y) \in \mathbb{R}_0^2. \quad (3.3)$$

Note that h is continuous on \mathbb{R}_0^2 and satisfies the scaling property: for any $\lambda > 0$

$$\lambda h(\lambda^{1/H_1} x, \lambda^{1/H_2} y) = h(x, y), \quad \forall (x, y) \in \mathbb{R}_0^2. \quad (3.4)$$

With h in (3.3) we associate a family of Gaussian RFs indexed by $\gamma > 0$, as follows. For $\gamma = \gamma_0 := H_1/H_2$, set

$$V_{\gamma_0}(x, y) := \int_{\mathbb{R}^2} \frac{(1 - e^{iux})(1 - e^{ivy})}{i^2 uv} \sqrt{h(u, v)} W(du, dv), \quad (x, y) \in \bar{\mathbb{R}}_+^2, \quad (3.5)$$

where $\{W(dx, dy); (x, y) \in \mathbb{R}^2\}$ is a standard complex-valued Gaussian noise, $\overline{W(dx, dy)} = W(-dx, -dy)$, with zero mean and variance $E|W(dx, dy)|^2 = dx dy$. Define

$$V_+(x, y) := \begin{cases} \int_{\mathbb{R}^2} \frac{(1 - e^{iux})(1 - e^{ivy})}{i^2 uv} |u|^{-H_1/2} W(du, dv), & H_1 < 1, \\ x \rho_1 c^{(1-H_1)/4} \int_{\mathbb{R}} \frac{1 - e^{ivy}}{iv} |v|^{-(H_1 H_2 - H_2)/2H_1} W_1(dv), & H_1 > 1, \end{cases} \quad (3.6)$$

and

$$V_-(x, y) := \begin{cases} \int_{\mathbb{R}^2} \frac{(1 - e^{iux})(1 - e^{ivy})}{i^2 uv} |v|^{-H_2/2} W(du, dv), & H_2 < 1, \\ y \rho_2 c^{-H_1/4H_2} \int_{\mathbb{R}} \frac{1 - e^{iux}}{iu} |u|^{-(H_1 H_2 - H_1)/2H_2} W_1(du), & H_2 > 1. \end{cases} \quad (3.7)$$

where $\{W(dx, dy)\}$ is as in (3.5) and $\{W_1(dx); x \in \mathbb{R}\}$ is a standard complex-valued Gaussian noise on \mathbb{R} , $\overline{W_1(dx)} = W_1(-dx)$, with zero mean and variance $E|W_1(dx)|^2 = dx$, $\rho_1^2 := B(1/2, (H_1 - 1)/2)$, $\rho_2^2 := (H_1/2H_2)B(H_1/H_2, (H_1 H_2 - H_1)/2H_2)$. Here and below, $B(\cdot, \cdot)$ is the beta function. We also define

$$H_+(\gamma) := \begin{cases} (1 + \gamma + H_1)/2, & H_1 < 1, \\ (\gamma H_1 + \gamma H_1 H_2 - \gamma H_2 + 2H_1)/2H_1, & H_1 > 1, \end{cases} \quad (3.8)$$

$$H_-(\gamma) := \begin{cases} (1 + \gamma + \gamma H_2)/2, & H_2 < 1, \\ (H_2 + H_1 H_2 - H_1 + 2\gamma H_2)/2H_2, & H_2 > 1, \end{cases} \quad (3.9)$$

where $\gamma > 0$, and

$$H(\gamma_0) := (H_1 + H_2 + H_1 H_2)/2H_2, \quad H_1 \neq 1, H_2 \neq 1. \quad (3.10)$$

Note $H_+(\gamma_0) = H_-(\gamma_0) = H(\gamma_0)$.

Proposition 3.1 (i) The RFs $V_{\gamma_0} = \{V_{\gamma_0}(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$, $V_+ = \{V_+(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ and $V_- = \{V_-(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ in (3.5)-(3.7) are well-defined for any $0 < H_1 \leq H_2 < \infty$, $H_1 H_2 < H_1 + H_2$, $c > 0$ with exception of $H_1 = 1$ in (3.6) and $H_2 = 1$ in (3.7). These RFs have zero mean, finite variance and stationary rectangular increments in the sense of (2.1). Furthermore,

$$V_+ \stackrel{\text{fdd}}{=} \kappa_{H_1, H_2}^+ \begin{cases} B_{(H_1+1)/2, 1/2} & H_1 < 1, \\ B_{1, (1+H_2-H_2/H_1)/2}, & H_1 > 1, \end{cases} \quad (3.11)$$

$$V_- \stackrel{\text{fdd}}{=} \kappa_{H_1, H_2}^- \begin{cases} B_{1/2, (H_2+1)/2} & H_2 < 1, \\ B_{(1+H_1-H_1/H_2)/2, 1}, & H_2 > 1, \end{cases} \quad (3.12)$$

where B_{H_1, H_2} is a fractional Brownian sheet (see Example 2.2), and $\kappa_{H_1, H_2}^\pm > 0$ are some constants.

(ii) V_{γ_0}, V_+, V_- are operator scaling RFs: for any $\lambda > 0$,

$$\{V_{\gamma_0}(\lambda x, \lambda^{\gamma_0} y); (x, y) \in \bar{\mathbb{R}}_+^2\} \stackrel{\text{fdd}}{=} \{\lambda^{H(\gamma_0)} V_{\gamma_0}(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}, \quad (3.13)$$

$$\{V_+(\lambda x, \lambda^\gamma y); (x, y) \in \bar{\mathbb{R}}_+^2\} \stackrel{\text{fdd}}{=} \{\lambda^{H_+(\gamma)} V_+(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}, \quad (3.14)$$

$$\{V_-(\lambda x, \lambda^\gamma y); (x, y) \in \bar{\mathbb{R}}_+^2\} \stackrel{\text{fdd}}{=} \{\lambda^{H_-(\gamma)} V_-(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}, \quad (3.15)$$

where $H_+(\gamma), H_-(\gamma), H(\gamma_0)$ are defined in (3.8), (3.9), (3.10), respectively. In (3.14) and (3.15), $\gamma > 0$ is arbitrary.

(iii) V_{γ_0} has dependent rectangular increments, while V_+ and V_- have either independent, or invariant rectangular increments along one of the coordinate axes.

Proof. (i) Let us show that $V_\gamma(x, y)$ is well-defined as stochastic integral w.r.t. Gaussian white noise. It suffices to consider the case $x = y = 1$ only since the general case is analogous. Let $\gamma = \gamma_0$. Then

$$\begin{aligned} \text{EV}_{\gamma_0}^2(1, 1) &= \int_{\mathbb{R}^2} |1 - e^{iu}|^2 |1 - e^{iv}|^2 \frac{h(u, v)}{|uv|^2} du dv \leq C \int_0^\infty \int_0^\infty \frac{du dv}{(1+u^2)(1+v^2)(u^2 + v^{2H_2/H_1})^{H_1/2}} \\ &= C \left(\int_0^\infty du \int_0^1 dv \dots + \int_0^\infty du \int_1^\infty dv \dots \right) =: C(J_1 + J_2). \end{aligned}$$

By change of variable $v = u^{H_1/H_2} z$, $J_1 \leq C \int_0^\infty \frac{du}{1+u^2} \int_0^{1/u^{H_1/H_2}} \frac{u^{H_1/H_2} dz}{u^{H_1(1+z^{2H_2/H_1})^{H_1/2}}} =: J'_1$. Let $H_2 > 1$. Then $J'_1 \leq C \int_0^\infty u^{(H_1/H_2)-H_1} (1+u^2)^{-1} du < \infty$ since $H_2(H_1 - 1) < H_1$. Next, let $H_2 < 1$. Then $J'_1 \leq C \int_0^\infty (1+u^2)^{-1} du < \infty$ and $J_1 < \infty$. The case $H_1 = 1$ follows similarly. The convergence $J_2 < \infty$ is obvious.

Let us show that (3.6) is well-defined. Let $H_1 < 1$, then $\text{EV}_\gamma^2(1, 1) \leq C \int_{\mathbb{R}_+^2} du dv (1+u^2)^{-1} (1+v^2)^{-1} v^{-H_1} < \infty$. Next, let $H_1 > 1$, then $\text{EV}_\gamma^2(1, 1) \leq C \int_0^\infty dv (1+v^2)^{-1} v^{-(H_1 H_2 - H_2)/H_1} < \infty$ since $H_2(H_1 - 1) < H_1$. The convergence of the stochastic integral in (3.7) follows in a similar way.

Let $K_{(x,y);(x',y')}, (x, y) \preceq (x', y')$ be a rectangle in \mathbb{R}_+^2 . Then from (3.5) we immediately obtain $V_{\gamma_0}(K_{(x,y);(x',y')}) = \tilde{V}_{\gamma_0}(K_{(0,0);(x'-x,y'-y)})$, where $\tilde{V}_{\gamma_0}(x, y)$ is defined as in (3.5) with $W(du, dv)$ replaced by $\tilde{W}(du, dv) := e^{i(ux+vy)} W(du, dv)$. Clearly $\{\tilde{W}(du, dv); (u, v) \in \mathbb{R}^2\} \stackrel{\text{fdd}}{=} \{W(du, dv); (u, v) \in \mathbb{R}^2\}$ for any $(x, y) \in \mathbb{R}_+^2$. Hence, V_{γ_0} has stationary rectangular increments. The same fact for V_\pm follows analogously.

Relation (3.11) follows from Gaussianity and

$$\begin{aligned} & \mathbb{E}[V_+(x, y)V_+(x', y')] \\ &= (\kappa_{H_1, H_2}^+)^2 \begin{cases} \mathbb{E}[B_{(H_1+1)/2}(x)B_{(H_1+1)/2}(x')]\mathbb{E}[B_{1/2}(y)B_{1/2}(y')], & H_1 < 1, \\ \mathbb{E}[B_1(x)B_1(x')]\mathbb{E}[B_{(1+H_2-H_2/H_1)/2}(y)B_{(1+H_2-H_2/H_1)/2}(y')], & H_1 > 1 \end{cases} \end{aligned} \quad (3.16)$$

for any $x, x', y, y' \geq 0$. Let $H_1 < 1$, then the l.h.s. of (3.16) factorizes as the product of two integrals $\int_{\mathbb{R}} (1 - e^{iux})(1 - e^{-iux'})|u|^{-2-H_1} du \int_{\mathbb{R}} (1 - e^{ivy})(1 - e^{-ivy'})|v|^{-2} dv$, equal to the covariances on the r.h.s., see e.g. [30], Prop.9.2. The case $H_1 > 1$ in (3.16) and (3.12) are analogous.

(ii) The operator scaling property follows from scaling properties of the integrands, see (3.4), and the white noise, viz., $\{W(du/\lambda, dv/\lambda^\gamma); (u, v) \in \mathbb{R}^2\} \stackrel{\text{fdd}}{=} \{\lambda^{-(1+\gamma)/2}W(du, dv); (u, v) \in \mathbb{R}^2\}$, $\{W_1(du/\lambda); u \in \mathbb{R}\} \stackrel{\text{fdd}}{=} \{\lambda^{-1/2}W_1(du); u \in \mathbb{R}\}$.

(iii) The fact that V_+, V_- do not have dependent rectangular increments follows from (3.11), (3.12) and the properties of fractional Brownian sheet stated in Example 2.2. The proof that V_{γ_0} in (3.5) has dependent increments (i.e., neither independent nor invariant rectangular increments in any direction) is part of a more general statement in Lemma 3.1 below. Proposition 3.1 is proved. \square

Theorem 3.1 *Let X be a stationary zero-mean Gaussian RF on \mathbb{Z}^2 with zero mean and spectral density f in (3.2), where $c > 0$, $0 < H_1 \leq H_2 < \infty$, $H_1, H_2 \neq 1$ and $H_1 H_2 < H_1 + H_2$. Then for any $\gamma > 0$ the limit of partial sums*

$$n^{-H(\gamma)} S_{n\gamma}(x, y) \xrightarrow{\text{fdd}} V_\gamma(x, y), \quad (x, y) \in \mathbb{R}_+^2, \quad n \rightarrow \infty \quad (3.17)$$

exists with

$$V_\gamma := \begin{cases} V_+, & \gamma > \gamma_0, \\ V_-, & \gamma < \gamma_0, \\ V_{\gamma_0}, & \gamma = \gamma_0, \end{cases}, \quad H(\gamma) := \begin{cases} H_+(\gamma), & \gamma > \gamma_0, \\ H_-(\gamma), & \gamma < \gamma_0, \\ H(\gamma_0), & \gamma = \gamma_0, \end{cases} \quad (3.18)$$

and $V_\pm, V_{\gamma_0}, H_\pm(\gamma), H(\gamma_0)$ given in (3.5)-(3.7) and (3.8)-(3.10), respectively. As a consequence, the RF X exhibits scaling transition at $\gamma_0 = H_1/H_2$. Moreover, X has Type I isotropic distributional LRD if $H_1 = H_2$ and Type I anisotropic distributional LRD if $H_1 \neq H_2$.

Remark 3.1 The existence of the limit in (3.17) in the cases $\gamma > \gamma_0, H_1 = 1$ and $\gamma < \gamma_0, H_2 = 1$ is an open question. Note that V_+ in (3.6) is undefined for $H_1 = 1$ and, similarly, V_- in (3.7) is undefined for $H_2 = 1$. Nevertheless, relations (3.11)-(3.12) suggest that the limit (3.17) might exist also in the above cases and be given by $V_+ = \kappa_+ B_{1,1/2}$ and $V_- = \kappa_- B_{1/2,1}$ for some constants $\kappa_\pm \neq 0$.

Proof of Theorem 3.1. Let us prove the convergence in (3.17). Recall the definition of $S_{n\gamma}(x, y)$ in (3.1). By Gaussianity, this follows from

$$R_{n\gamma}(x, y; x', y') := n^{-2H(\gamma)} \mathbb{E}[S_{n\gamma}(x, y)S_{n\gamma}(x', y')] \rightarrow \mathbb{E}[V_\gamma(x, y)V_\gamma(x', y')], \quad n \rightarrow \infty. \quad (3.19)$$

We have, with $m := n^\gamma$, and $D_n(u) := \sum_{t=1}^n e^{itu} = (e^{iu} - e^{i(n+1)u})/(1 - e^{iu})$, $|u| < \pi$,

$$R_{n\gamma}(x, y; x', y') = n^{-2H(\gamma)} \int_{\Pi^2} D_{[nx]}(u) \overline{D_{[nx']}(u)} D_{[my]}(v) \overline{D_{[my']}(v)} f(u, v) du dv.$$

Consider (3.19) for $\gamma = \gamma_0$. By change of variables, $R_n^\gamma(x, y; x', y') = \int_{\mathbb{R}^2} G_n(u, v) du dv$, where

$$G_n(u, v) := \frac{1}{n^2 m^2} D_{[nx]}(\frac{u}{n}) \overline{D_{[nx']}(u)} D_{[my]}(\frac{v}{m}) \overline{D_{[my']}(v)} n^{-H_1} f(\frac{u}{n}, \frac{v}{m}) \mathbf{1}(|u| \leq \pi n, |v| \leq \pi m) \quad (3.20)$$

and we used the fact that $nmn^{H_1} = n^{2H(\gamma_0)}$. From (3.3) it follows that $n^{-H_1} f(\frac{u}{n}, \frac{v}{m}) \rightarrow h(u, v)$ a.e. in \mathbb{R}^2 and hence

$$G_n(u, v) \rightarrow G(u, v) := \left(\frac{1 - e^{ixu}}{iu} \right) \left(\frac{1 - e^{-ix'u}}{-iu} \right) \left(\frac{1 - e^{iyv}}{iv} \right) \left(\frac{1 - e^{-iy'v}}{-iv} \right) h(u, v) \quad \text{a.e. in } \mathbb{R}^2,$$

where $\int_{\mathbb{R}^2} G(u, v) du dv = E[V_{\gamma_0}(x, y) V_{\gamma_0}(x', y')]$. Moreover, $|n^{-1} D_{[nx]}(\frac{u}{n})| \leq Cx(1 + |([nx]/n)u|) \leq C/(1 + |u|)$, $|u| < n\pi$ for any fixed $x \in \mathbb{R}, x \neq 0$. Together with $f(u, v) \leq Ch(u, v)$, see (3.2), (3.3), this implies for any fixed $x, x', y, y' > 0$ that

$$|G_n(u, v)| \leq C(1 + u^2)^{-1}(1 + v^2)^{-1} h(u, v) =: \bar{G}(u, v),$$

where $\int_{\mathbb{R}^2} \bar{G}(u, v) du dv < \infty$ (see above). Therefore, (3.19) for $\gamma = \gamma_0$ follows by the dominated convergence theorem.

Consider (3.19) for $\gamma > \gamma_0, 0 < H_1 < 1$. We have again $R_{n\gamma}(x, y; x', y') = \int_{\mathbb{R}^2} G_n(u, v) du dv$ with G_n given in (3.20) and $n^{-H_1} f(\frac{u}{n}, \frac{v}{m}) \rightarrow h(u, 0) = |u|^{-H_1}$ a.e. in \mathbb{R}^2 and $n^{-H_1} f(\frac{u}{n}, \frac{v}{m}) \leq Ch(u, 0), u \in \mathbb{R}$. Since $\bar{G}(u, v) := C(1 + u^2)^{-1}(1 + v^2)^{-1} h(u, 0)$ is integrable on \mathbb{R}^2 for $0 < H_1 < 1$, this proves (3.19).

Consider (3.19) for $\gamma > \gamma_0, H_1 > 1$. Then

$$\begin{aligned} R_{n\gamma}(x, y; x', y') &= \int_{\mathbb{R}^2} \tilde{G}_m(u, v) du dv, \quad \text{with} \\ \tilde{G}_m(u, v) &:= L_{m1}(u) L_{m2}(v) L_{m3}(u, v) L_4(u, v) \mathbf{1}(|u| \leq \pi m^{1/\gamma_0}, |v| \leq \pi m), \end{aligned}$$

where $m = n^\gamma$, $m^{(1/\gamma)-(1/\gamma_0)} \rightarrow 0$ and

$$\begin{aligned} L_{m1}(u) &:= \frac{n^{2((\gamma/\gamma_0)-1)}(1 - e^{iu([nx]/n^{\gamma/\gamma_0})})(1 - e^{-iu([nx']/n^{\gamma/\gamma_0})})}{|n^{\gamma/\gamma_0}(1 - e^{iu/n^{\gamma/\gamma_0}})|^2} \rightarrow xx', \\ L_{m2}(v) &:= \frac{(1 - e^{iv([my]/m)})(1 - e^{-iv([my']/m)})}{|m(1 - e^{iv/m})|^2} \rightarrow \frac{(1 - e^{ivy})(1 - e^{-ivy'})}{v^2}, \\ L_{m3}(u, v) &:= g(\frac{u}{m^{1/\gamma_0}}, \frac{v}{m}) \rightarrow 1, \\ L_4(u, v) &:= \frac{1}{(u^2 + cv^{2/\gamma_0})^{H_1/2}}, \end{aligned} \quad (3.21)$$

as $m \rightarrow \infty$. Note that for fixed $x, x', y, y' > 0$, all three convergences in (3.21) are uniform in $(u, v) \in \mathbb{R}^2$ on each compact set in \mathbb{R}^2 , the limit functions being bounded and continuous in \mathbb{R}^2 , moreover, $L_{mi}, i = 1, 2, 3$ are bounded on $|u| \leq \pi m^{1/\gamma_0}, |v| \leq \pi m$. Therefore, by the dominated convergence theorem, as $n \rightarrow \infty$,

$$\begin{aligned} R_{n\gamma}(x, y; x', y') &\rightarrow xx' \int_{\mathbb{R}} \frac{(1 - e^{ivy})(1 - e^{-ivy'})}{v^2} dv \int_{\mathbb{R}} \frac{du}{(u^2 + cv^{2/\gamma_0})^{H_1/2}} \\ &= \rho_1^2 c^{(1-H_1)/2} xx' \int_{\mathbb{R}} \frac{(1 - e^{ivy})(1 - e^{-ivy'})}{v^2} \frac{du}{|v|^{(H_1 H_2 - H_2)/H_1}}, \end{aligned}$$

where $\rho_1^2 = \int_{\mathbb{R}} \frac{du}{(u^2+1)^{H_1/2}} = B(1/2, (H_1 - 1)/2)$. The last limit agrees with the covariance on the r.h.s. of (3.19), see the definition of $V_+ = V_\gamma$ in (3.6), proving (3.19) for $\gamma > \gamma_0$. The proof of (3.19) for $\gamma < \gamma_0$ is analogous. This proves the convergence in (3.17). The second statement of the theorem follows from Proposition 3.1. \square

Next, we consider Gaussian RFs with spectral density $f = f_\Pi$ in (1.3):

$$f(x, y) = \frac{g(x, y)}{|x|^{2d_1}|y|^{2d_2}}, \quad (x, y) \in \Pi^2. \quad (3.22)$$

Proposition 3.2 *Let X be a stationary Gaussian RF on \mathbb{Z}^2 with zero mean and spectral density f in (3.22), where $0 < d_1, d_2 < 1/2$ and $g \geq 0$ is a bounded function such that $\lim_{x,y \rightarrow 0} g(x, y) = 1$. Then for any $\gamma > 0$*

$$n^{-H(\gamma)} S_{n\gamma}(x, y) \xrightarrow{\text{fdd}} \kappa(d_1)\kappa(d_2)B_{d_1+.5, d_2+.5}(x, y), \quad (x, y) \in \mathbb{R}_+^2, \quad n \rightarrow \infty, \quad (3.23)$$

where $H(\gamma) := (1 + \gamma)/2 + d_1 + d_2\gamma$ and $B_{d_1+.5, d_2+.5}$ is a fractional Brownian sheet (see Example 2.2 for definition), $\kappa^2(d) := \int_{\mathbb{R}} |1 - e^{ix}|^2 |x|^{-2-2d} dx = \pi(2(d + .5)^2 \Gamma(d) \cos(\pi d))^{-1}$. As a consequence, X does not exhibit scaling transition for any $0 < d_1, d_2 < 1/2$.

Proof. We follow the proof of Theorem 3.1. Accordingly, it suffices to show (3.19), where

$$V_\gamma(x, y) := \int_{\mathbb{R}^2} \frac{(1 - e^{iux})(1 - e^{ivy})}{i^2 uv} \sqrt{h(u, v)} W(du, dv), \quad \text{with } h(u, v) := |u|^{-2d_1} |v|^{-2d_2}, \quad (3.24)$$

is the spectral representation of fractional Brownian sheet, viz., $V_\gamma \stackrel{\text{fdd}}{=} \kappa(d_1)\kappa(d_2)B_{d_1+.5, d_2+.5}$, see ([22], (7)) and $W(du, dv)$ is the same as in (3.5). Note the r.h.s. of (3.24) does not depend on γ . Let $m = n^\gamma$. Then

$$\begin{aligned} & R_{n\gamma}(x, y; x', y') \\ &= n^{-2H(\gamma)} \int_{\Pi^2} D_{[nx]}(u) \overline{D_{[nx']}(u)} D_{[my]}(v) \overline{D_{[my']}(v)} |u|^{-2d_1} |v|^{-2d_2} g(u, v) du dv \\ &\sim n^{-1-2d_1} \int_{\Pi} D_{[nx]}(u) \overline{D_{[nx']}(u)} |u|^{-2d_1} du \, m^{-1-2d_2} \int_{\Pi} D_{[my]}(v) \overline{D_{[my']}(v)} |v|^{-2d_2} dv. \end{aligned}$$

The limit $\lim_{n \rightarrow \infty} n^{-1-2d} \int_{\Pi} D_{[nx]}(u) \overline{D_{[nx']}(u)} |u|^{-2d} du = (1/2)\kappa^2(d)(x^{2d+1} + x'^{2d+1} - |x - x'|^{2d+1})$, $0 < d < 1/2$, $x, x' > 0$ is well-known. This proves (3.19) and the proposition, too. \square

Given a line $\ell = \{ax + by = 0\} \subset \mathbb{R}^2$, a function $k(u, v), (u, v) \in \mathbb{R}_0^2$ is said ℓ -degenerated if $k(u, v) = \tilde{k}(au + bv), (u, v) \in \mathbb{R}_0^2$, where $\tilde{k}(z)$ is a function of a single variable $z \in \mathbb{R}$. Obviously, $h(u, v)$ in (3.3) is not ℓ -degenerated for any ℓ . The proof of Lemma 3.1 is given in sec.4.

Lemma 3.1 *Let*

$$V(x, y) = \int_{\mathbb{R}^2} \frac{(1 - e^{iux})(1 - e^{ivy})}{i^2 uv} \sqrt{k(u, v)} W(du, dv), \quad (x, y) \in \bar{\mathbb{R}}_+^2, \quad (3.25)$$

be a Gaussian RF, where $W(du, dv)$ is the same as in (3.5) and $k(u, v) \geq 0$ is a measurable function such that

$$\int_{\mathbb{R}^2} \frac{k(u, v) du dv}{(1 + u^2)(1 + v^2)} < \infty. \quad (3.26)$$

Let ℓ be a line in \mathbb{R}^2 . Then $V = \{V(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ has independent rectangular increments in direction ℓ if and only if k is ℓ -degenerated. Moreover, V does not have invariant rectangular increments in any direction.

The following proposition obtains scaling transition of different type than Type I for the class of stationary Gaussian RFs with spectral density

$$f(x, y) = g(x, y)|ax + by|^{-2d}, \quad (3.27)$$

where $0 < d < 1/2$, a, b are parameters and g is continuous at the origin. Notice that when $ab = 0$, (3.27) is the limiting case of Type II density in (3.22) when one of the parameters d_1, d_2 approaches zero. Partial sums limits of such RFs were discussed in Lavancier [17].

Proposition 3.3 *Let X be a stationary Gaussian RF on \mathbb{Z}^2 with zero mean and spectral density f in (3.27), where $0 < d < 1/2$, $ab \neq 0$ and $g \geq 0$ is a bounded function such that $\lim_{x,y \rightarrow 0} g(x, y) = 1$. Then for any $\gamma > 0$ the limit V_γ in (3.17) exists and is written as in (3.24) with $H(\gamma) > 0$ and $h(u, v) = h_\gamma(u, v)$ given by*

$$H(\gamma) := \begin{cases} (1 + \gamma)/2 + d, & \gamma \geq 1, \\ (1 + \gamma)/2 + \gamma d, & \gamma < 1, \end{cases} \quad h_\gamma(u, v) := \begin{cases} |au + bv|^{-2d}, & \gamma = 1, \\ |au|^{-2d}, & \gamma > 1, \\ |bv|^{-2d}, & \gamma < 1. \end{cases} \quad (3.28)$$

In particular, the RF X exhibits scaling transition at $\gamma_0 = 1$ with $V_+ = \kappa_+ B_{1/2+d, 1/2}$, $V_- = \kappa_- B_{1/2, 1/2+d}$, where $\kappa_\pm \neq 0$ are some constants. Moreover, X does not have Type I distributional LRD property.

Proof. Let $\gamma > 1$. Then similarly as in the proof of Theorem 3.1, case $\gamma > \gamma_0$, $0 < H_1 < 1$, with $m = n^\gamma$, we have that $R_{n^\gamma}(x, y; x', y') = \int_{\mathbb{R}^2} G_n(u, v) du dv$, where G_n is given in (3.20) and $n^{-2d} f(\frac{u}{n}, \frac{v}{m}) \rightarrow h_\gamma(u, v) = |au|^{-2d}$ a.e. in \mathbb{R}^2 ; moreover, G_n is dominated by integrable function $\bar{G}(u, v) := C(1 + u^2)^{-1}(1 + v^2)^{-1}|u|^{-2d}$. This proves (3.19) for $\gamma > 1$ and the proof in the remaining cases $\gamma = 1$ and $\gamma < 1$ is analogous. The fact that X admits scaling transition at $\gamma_0 = 1$ is obvious since $B_{1/2+d, 1/2} \stackrel{\text{fdd}}{\neq} c B_{1/2, 1/2+d}$ for any $c > 0$. Finally, since h_1 in (3.28) is degenerated, by Lemma 3.1 the RF V_1 has independent increments in the direction perpendicular to the line $ax + by = 0$ and therefore X does not have Type I distributional LRD property. Proposition 3.3 is proved. \square

Remark 3.2 The above result can be described as an abrupt change of the ‘dependence axis’ of RF X under unbalanced scaling. The form of spectral density in (3.27) suggests that the LRD in X is essentially ‘one-dimensional’ along the line $\ell = \{at + bs = 0\} \subset \mathbb{R}^2$. The ‘supercritical regime’ $\gamma > 1$ transforms this ‘dependence axis’ ℓ into the horizontal axis, since $V_+ = \kappa_+ B_{1/2+d, 1/2}$ has independent increments in the vertical direction. A similar transformation of ℓ into the vertical axis occurs in the ‘subcritical regime’ $\gamma < 1$.

Remark 3.3 As noted in [27], scaling transition occurs for a very different class of models under joint temporal and contemporaneous aggregation of *independent* LRD processes in telecommunication and economics, see [23], [14], [10], [25] and the references therein. In these works, $\{X(t, s); t \in \mathbb{Z}, s \in \mathbb{Z}\}$ are independent copies of a stationary LRD process $Y = \{Y(t); t \in \mathbb{Z}\}$ and the scaling limits V_γ of the RF $X = \{X(t, s); (t, s) \in \mathbb{Z}^2\}$ necessarily have independent increments in the vertical direction for any $\gamma > 0$, meaning that X cannot have

Type I distributional LRD by definition. Nevertheless for heavy-tailed centered ON/OFF process Y and some other duration based models, the results in [23] imply that the above RF X can exhibit scaling transition with some $\gamma_0 \in (0, 1)$ and markedly distinct ‘supercritical’ and ‘subcritical’ unbalanced scaling limits V_{\pm} , viz., V_+ being a Gaussian RF with dependent increments in the horizontal direction and V_- having α -stable ($1 < \alpha < 2$) distributions and independent increments in the horizontal direction. The well-balanced scaling limit V_{γ_0} in the above models was discussed in detail in [13], [25] and was shown to have interesting ‘intermediate’ properties between V_+ and V_- .

4 Proof of Lemma 3.1

We use some facts about generalized functions (Schwartz distributions) (see [31]). Let $S(\mathbb{R}^\nu)$ ($\nu = 1, 2$) be the Schwartz space of all rapidly decreasing C^∞ -functions $\phi : \mathbb{R}^\nu \rightarrow \mathbb{R}$, and $S'(\mathbb{R}^\nu)$ be the space of all generalized functions $T : S(\mathbb{R}^\nu) \rightarrow \mathbb{R}$. The Fourier transform $\widehat{T} \in S'(\mathbb{R}^2)$ of $T \in S'(\mathbb{R}^2)$ is defined as $\widehat{T}(\phi) = T(\widehat{\phi})$, $\phi \in S(\mathbb{R}^2)$, where $\widehat{\phi}(u, v) := \int_{\mathbb{R}^2} e^{i(ux+vy)} \phi(x, y) dx dy$. For $\varphi, \psi \in S(\mathbb{R})$, denote $(\varphi \otimes \psi)(u, v) := \varphi(u)\psi(v)$, $(\varphi \otimes \psi) \in S(\mathbb{R}^2)$. Note that for any rectangle K we have $V(K) = \int_{\mathbb{R}^2} \widehat{\mathbf{1}}_K(u, v) \sqrt{h(u, v)} W(du, dv)$, where $\mathbf{1}_K$ is the indicator function of K . It is easy to show that $\{V(x, y)\}$ in (3.25) extends to a generalized stationary Gaussian random field ([8]):

$$\mathcal{V}(\phi) := \int_{\mathbb{R}^2} \widehat{\phi}(u, v) \sqrt{k(u, v)} W(du, dv), \quad \phi \in S(\mathbb{R}^2).$$

Let ℓ be a given line and $\mathbb{R}_{\pm}^2(\ell')$ be the open halfplanes separated by line $\ell' \perp \ell$, $0 \in \ell'$, viz., $\mathbb{R}_{\pm}^2(\ell') \cup \ell' \cup \mathbb{R}_{\mp}^2(\ell') = \mathbb{R}^2$. Let us show that the statements (a) $\{V(x, y); (x, y) \in \bar{\mathbb{R}}_{\pm}^2\}$ has independent rectangular increments in direction ℓ and (b) $\mathcal{V}(\phi_+)$ and $\mathcal{V}(\phi_-)$ are independent for any $\phi_{\pm} \in S(\mathbb{R}^2)$ with supports in $\mathbb{R}_{\pm}^2(\ell')$ are equivalent. Statements (a) and (b) can be rewritten as

$$\int_{\mathbb{R}^2} \widehat{\phi}_+(u, v) \overline{\widehat{\phi}_-(u, v)} k(u, v) du dv = 0, \quad \phi_{\pm} \in L(\mathbb{R}_{\pm}^2(\ell')), \quad (4.1)$$

and

$$\int_{\mathbb{R}^2} \widehat{\phi}_+(u, v) \overline{\widehat{\phi}_-(u, v)} k(u, v) du dv = 0, \quad \phi_{\pm} \in S(\mathbb{R}^2), \quad \text{supp}(\phi_{\pm}) \subset \mathbb{R}_{\pm}^2(\ell'), \quad (4.2)$$

respectively, where $L(\mathbb{R}_{\pm}^2(\ell'))$ is the set of all linear combinations of indicator functions $\mathbf{1}_K$ of rectangles $K \subset \mathbb{R}_{\pm}^2(\ell')$. To show the implication (b) \Rightarrow (a), note that any indicator functions $\mathbf{1}_{K_{\pm}}, K_{\pm} \subset \mathbb{R}_{\pm}^2(\ell')$ can be approximated in $L^2(\mathbb{R}^2)$ by elements $\phi_{\pm, \epsilon} \in S(\mathbb{R}^2)$, $\epsilon > 0$ with compact supports $\text{supp}(\phi_{\pm, \epsilon}) \subset \mathbb{R}_{\pm}^2(\ell')$. The approximating functions can be taken as $\phi_{\pm, \epsilon} = \mathbf{1}_{K_{\pm}} \star \theta_{\epsilon}$, $\epsilon > 0$, where $\theta_{\epsilon}(u, v) := \epsilon^{-2} \theta(u/\epsilon, v/\epsilon)$, θ is a $C_0^\infty(\mathbb{R}^2)$ probability kernel, and \star denotes the convolution. See [28], Thm.1.18. Using $|\widehat{\phi}_{\pm, \epsilon}(u, v)| = |\widehat{\mathbf{1}}_{K_{\pm}}(u, v)| |\widehat{\theta}_{\epsilon}(u, v)|$, $|\widehat{\mathbf{1}}_{K_{\pm}}(u, v)| \leq C(1 + |u|)^{-1}(1 + |v|)^{-1}$ and $|\widehat{\theta}_{\epsilon}(u, v)| = |\widehat{\theta}(\epsilon u, \epsilon v)| \leq C$, with $C < \infty$ independent of $\epsilon \rightarrow 0$, we can easily show that (4.2) implies (4.1) for $\phi_{\pm} = \mathbf{1}_{K_{\pm}}$ and any rectangles $K_{\pm} \subset \mathbb{R}_{\pm}^2(\ell')$, proving the implication (b) \Rightarrow (a).

Next, consider the converse implication (a) \Rightarrow (b). It suffices to prove (4.2) for $\phi_{\pm} \in S(\mathbb{R}^2)$ with compact supports $\text{supp}(\phi_{\pm}) \subset \mathbb{R}_{\pm}^2(\ell')$. Consider approximation of such ϕ_{\pm} by step functions $\phi_{\pm, n}(x, y) :=$

$\sum_{(k,j) \in \mathbb{Z}^2} \phi(k/n, j/n) \mathbf{1}_{(k/n, (k+1)/n] \times (j/n, (j+1)/n]}(x, y)$, $n \in \mathbb{N}_+$. Then $\sup_{x,y} |\phi_{\pm,n}(x, y) - \phi_{\pm}(x, y)| \rightarrow 0$ ($n \rightarrow \infty$) and $\phi_{\pm,n} \in L(\mathbb{R}_+^2(\ell'))$ for all $n > n_0$ large enough; moreover,

$$|\widehat{\phi}_{\pm,n}(u, v)| \leq C(1 + |u|)^{-1}(1 + |v|)^{-1} \quad (4.3)$$

with $C < \infty$ independent of n . Inequality (4.3) can be shown using summation by parts, as follows. We have $|\widehat{\phi}_{\pm,n}(u, v)| = |uv|^{-1} |\sum_{k,j} \psi_{\pm,n}(k, j) e^{i(ku/n)} e^{i(jv/n)}| = |uv|^{-1} |\sum_{k,j} \psi_{\pm,n}(k, j) (e^{i(ku/n)} - 1)(e^{i(jv/n)} - 1)|$, where $\psi_{\pm,n}(k, j) := \phi_{\pm}((k-1)/n, (j-1)/n) - \phi_{\pm}((k-1)/n, j/n) - \phi_{\pm}(k/n, (j-1)/n) + \phi_{\pm}(k/n, j/n)$ is the double difference satisfying $|\psi_{\pm,n}(k, j)| \leq C/n^2$ (recall that ϕ_{\pm} is infinitely differentiable with compact support). Hence, $|\widehat{\phi}_{\pm,n}(u, v)| \leq C|uv|^{-1}(1 \wedge |u|)(1 \wedge |v|) \leq C(1 + |u|)^{-1}(1 + |v|)^{-1}$, proving (4.3). The above facts together with condition (3.26) allow using the dominated convergence criterion to prove (4.2) from (4.1) via the above approximation by step functions.

Consider first the case when ℓ is the horizontal axis, in which case ℓ' is the vertical axis. Let k be ℓ -degenerated, or $k(u, v) = \tilde{k}(v)$ for some measurable function \tilde{k} satisfying (3.26). Let $\phi_{\pm} = \varphi_{\pm} \otimes \psi_{\pm}$, where $\varphi_{\pm}, \psi_{\pm} \in S(\mathbb{R})$ satisfy $\text{supp}(\varphi_+) \subset (0, \infty), \text{supp}(\varphi_-) \subset (-\infty, 0)$. Then $E[\mathcal{V}(\phi_+)\mathcal{V}(\phi_-)] = J_1 J_2 = 0$ follows from

$$J_1 := \int_{\mathbb{R}} \widehat{\psi}_+(v) \overline{\widehat{\psi}_-(v)} \tilde{k}(v) dv, \quad J_2 := \int_{\mathbb{R}} \widehat{\phi}_+(u) \overline{\widehat{\phi}_-(u)} du = 2\pi \int_{\mathbb{R}} \phi_+(z) \phi_-(z) dz = 0$$

by Parseval's identity. Then (4.2) follows by taking linear combinations of the above ϕ_{\pm} in a standard way, proving that $\{V(x, y); (x, y) \in \mathbb{R}_+^2\}$ has independent rectangular increments in the horizontal ℓ .

Let us prove the converse implication, i.e. that (4.2) implies that k is ℓ -degenerated. Since the bilinear form $\mathcal{T}(\hat{\phi}_+, \hat{\phi}_-) := \int_{\mathbb{R}^2} \hat{\phi}_+(u, v) \overline{\hat{\phi}_-(u, v)} k(u, v) du dv$ in (4.2) is invariant with respect to shifts of ϕ_{\pm} in the horizontal direction, it is easy to see that (4.2) holds for any $\phi_{\pm} \in S(\mathbb{R}^2)$ with

$$\text{supp}(\phi_+) \subset (\mathbb{R} \setminus [-\epsilon, \epsilon]) \times \mathbb{R}, \quad \text{supp}(\phi_-) \subset (-\epsilon, \epsilon) \times \mathbb{R}, \quad \epsilon > 0. \quad (4.4)$$

Let $\phi_+ = \varphi \otimes \psi_+, \phi_- = \phi_{\epsilon} \otimes \psi_-$, where $\varphi, \phi_{\epsilon}, \psi_{\pm} \in S(\mathbb{R})$ satisfy $\text{supp}(\varphi) \subset \mathbb{R} \setminus \{0\}$ and $\phi_{\epsilon}(x) = \epsilon^{-1} \phi(x/\epsilon)$, where $\phi \in S(\mathbb{R})$ is a symmetric probability density with $\text{supp}(\phi) \subset (-1, 1)$. Note ϕ_{\pm} satisfy (4.4) for $\epsilon > 0$ small enough, implying

$$T_{\epsilon}(\hat{\varphi}) := \mathcal{T}(\hat{\varphi} \otimes \hat{\psi}_+, \hat{\phi}_{\epsilon} \otimes \hat{\psi}_-) = 0, \quad \text{supp}(\varphi) \subset \mathbb{R} \setminus \{0\} \quad (4.5)$$

according to (4.2). We claim that $\lim_{\epsilon \rightarrow 0} T_{\epsilon}(\hat{\varphi}) = T(\hat{\varphi})$, where the generalized function $T \in S'(\mathbb{R})$ is given by

$$T(\varphi) := \int_{\mathbb{R}} \varphi(u) \mathcal{K}(u) du, \quad \mathcal{K}(u) := \int_{\mathbb{R}} \hat{\psi}_+(v) \overline{\hat{\psi}_-(v)} k(u, v) dv. \quad (4.6)$$

Indeed, $|T_{\epsilon}(\hat{\varphi}) - T(\hat{\varphi})| \leq \int_{\mathbb{R}^2} |\hat{\varphi}(u)| |\hat{\phi}_{\epsilon}(u) - 1| |\hat{\psi}_+(v)| |\hat{\psi}_-(v)| |k(u, v)| du dv \rightarrow 0$ when $\epsilon \rightarrow 0$, which follows from $\hat{\phi}_{\epsilon}(u) \rightarrow 1$ and the integrability of $\hat{\varphi}(u) \hat{\psi}_+(v) \hat{\psi}_-(v) k(u, v)$ on \mathbb{R}^2 ; see (3.26), proving the above claim. Hence and from (4.5), we infer that $\hat{T}(\varphi) = T(\hat{\varphi})$ vanishes for φ with $\text{supp}(\varphi) \subset \mathbb{R} \setminus \{0\}$. The last fact implies that \hat{T} is a linear combination of Dirac's δ -function and its derivatives: $\hat{T}(\varphi) = \sum_{k=0}^p c_k \varphi^{(k)}(0)$, $\varphi^{(k)}(x) := d^k \varphi(x)/dx^k$ with real coefficients $c_k, 0 \leq k \leq p < \infty$, see ([31], Ch.1.13, Thm.3). In other words, $T(\varphi) =$

$(2\pi)^{-1} \sum_{k=0}^p c_k \hat{\varphi}^{(k)}(0) = \int_{\mathbb{R}} P(u) \varphi(u) du$, where $P(u) = (2\pi)^{-1} \sum_{k=0}^p c_k u^k$ is a polynomial of degree p . Comparing the last expression with (4.6) we obtain that $\mathcal{K}(u) = P(u)$ a.e. in \mathbb{R} . Since $|\mathcal{K}(u)| \leq C \int_{\mathbb{R}} (1 + v^2) k(u, v) dv$ satisfies $\int_{\mathbb{R}} |\mathcal{K}(u)| (1 + u^2)^{-1} du < \infty$, see (3.26), this means that the function \mathcal{K} in (4.6) is constant on the real line: $\mathcal{K}(u) = (2\pi)^{-1} c_0$. Since the last fact holds for arbitrary $\psi_{\pm} \in S(\mathbb{R})$, we conclude that the function k does not depend on u , viz., $k(u, v) = \tilde{k}(v)$ a.e. in \mathbb{R}^2 . This proves the first statement of the lemma for a horizontal line ℓ .

The case of a general line $\ell = \{au + bv = 0\}$ can be reduced to that of the horizontal line $\ell_1 := \{v = 0\}$ by a rotation of the plane. Indeed, in such a case, similarly as above we can show that there exists an orthogonal 2×2 -matrix O mapping ℓ to ℓ_1 such that $T_O(\hat{\varphi}) := \int_{\mathbb{R}} \hat{\varphi}(u) \mathcal{K}_O(u) du$ vanishes for φ with $\text{supp}(\varphi) \subset \mathbb{R} \setminus \{0\}$, where \mathcal{K}_O is defined as in (4.6) with $k(u, v)$ replaced by $k_O(u, v) := k(O^{-1}(u, v))$, and consequently $k_O(u, v)$ is ℓ_1 -degenerated, or $k(u, v)$ is ℓ -degenerated.

Let us prove the second statement of the lemma. Assume *ad absurdum* that $\{V(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ has invariant rectangular increments in the horizontal direction. Then $\mathcal{T}(\hat{\theta}_a \phi, \hat{\phi}) = \mathcal{T}(\hat{\theta}_0 \phi, \hat{\phi})$ does not depend on $a \in \mathbb{R}$, where $\theta_a \phi(x, y) := \phi(x + a, y)$ is a shifted function. Since $\mathcal{T}(\hat{\theta}_a \phi, \hat{\phi}) = \int_{\mathbb{R}^2} e^{iau} |\hat{\phi}(u, v)|^2 k(u, v) du dv \rightarrow 0$ ($a \rightarrow \infty$) by the Lebesgue theorem, we obtain a contradiction. The case when $\{V(x, y); (x, y) \in \mathbb{R}_+^2\}$ has invariant rectangular increments in arbitrary direction can be treated analogously. Lemma 3.1 is proved.

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